Original Article

# Characterization of Graph Families Using Distance Degree Sequence

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## Abstract

The shortest path or length between any two nodes or vertices of the graph $G$ is defined as the distance between those two vertices. The number of vertices that are adjacent to a particular vertex is defined to be the degree of that particular vertex. The Distance Degree Sequence of vertex v is $dds(v) = (d_0(v), d_1(v), \dots, d_{e(v)}(v))$ where $d_i(v)$ is the number of nodes or vertices at distance $i$ from $v$. $e(v)$ is called the eccentricity of the vertex $v$. The distance degree sequence of a vertex is usually denoted as dds. Here, we have discussed the construction and characterization of complete graphs, cycles of odd and even lengths, and complete bipartite graph $K_{2,m}$ using their distance degree sequence. Also, algorithms have been given for some of the results.

**Keywords:** Complete Graphs, Cycle Graphs, Complete Bipartite Graph, Distance Degree Sequence, Distance Degree Regular Graphs.

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## INTRODUCTION

In Graph Theory, the study of sequences is very common. A sequence, in general, is a list of enumeration of numbers. We prefer representing a graph with a sequence compared to just one number. This is because a sequence gives more information about a graph than an invariant. Many sequences are employed or used to represent graphs in the literature of graph theory. Some of them are the *degree sequence, path degree sequence, distance degree sequence, eccentric sequence, status sequence* and so on[10]. Let S be a sequence representing a graph. This sequence is said to be *graphical* if there is a graph that realizes the sequence. That is, for the given sequence, if there is a graph that accepts the sequence then the sequence is said to be a *graphical sequence.* We observe that for a graph it is easy to find a sequence that represents the graph, say any sequence mentioned above. But the converse part is too tricky[2]. That is, given a sequence, will there exist a graph that realizes the sequence? This leads to the concept of characterizing and constructing graphs or graph families using a given sequence. The realization of any sequence for a given graph was a primary question in the study of sequences of graphs.

Degree sequences were the the first type of sequences to be developed and studied. Erdos and Gallai[1] gave an existential characterization of graphs whereas the constructive characterization was given by Havel and Hakimi[2], [3]. Degree sequences find many applications including message transfer[15]. Degree sequence is applied in split graphs and such graphs are characterized. Then, Eccentric Sequences were the first to be conceptualized and researched[5]. Many results in this direction are applied in practical problems[6]. Later Nandakumar studied the minimal eccentric sequence[7]. The eccentric sequence for digraphs were developed much later in 2008 by Gilbert and Lopez[8]. Sequences based on the distance were studied such as the path degree sequence and distance degree sequence. Distance Degree sequence of a vertex is the number of vertices at distance 0, 1, 2, 3,…from a particular vertex of the graph. Randic, studied these sequences so that chemical isomers can be differentiated using their graphical structure[9]. Kennedy and Quintas studied this sequence to analyze the embedding of trees in lattice graphs and other spaces[14]. Many new results have been determined based on the distance degree sequence of graphs. Distance degree sequence is predominantly applied in the field of chemistry.

Path degree sequences are very helpful in describing atomic environments. The fundamental studies in the distance degree sequence later lead to various developments including embedding of graphs.

## PRELIMINARIES

We will now define all the terms that are necessary for proving the results in this paper.

Let us consider G to be the graph. V and E are considered as the vertex or node set and edge set respectively.

### Definition 2.1

For two vertices x, y in a graph, the *distance* between these two vertices, denoted as *d(x, y)* from x to y is defined to be the shortest $x - y$ path length.

### Definition 2.2

*Degree Sequence* of a graph is the enumeration of the list of number of edges that are adjacent to every vertex of the graph.

### Definition 2.3

Listing of the number of vertices at 1, 2, 3, …e(v) distances for a vertex say u in G is defined as the Distance Degree Sequence of that vertex u. Here, e(v) is the eccentricity of v in G. The distance degree sequence of vertex $v$ in G is considered as a sequence $\{d_{i0}, d_{i1}, d_{i2}, \dots d_{ij}, \dots\}$ where $d_{ij}$ is the number of nodes at j distance from $v$.

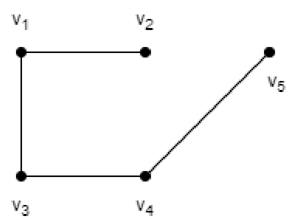


Fig. (i) Graph G

The Distance Degree Sequence dds(G) = ((1, 2, 1, 1), (1, 1, 1, 1, 1), (1, 2, 2), (1, 2, 1, 1), ( 1, 1, 1, 1, 1)). The distance degree sequence is abbreviated as dds in this paper for convenience.

### Definition 2.4

*Distance Degree Regular* (DDR) graph has same distance degree sequence(dds) for every vertex of the graph[11].

### Definition 2.5

If every vertex $v_{i,}\ i = 1,2, \dots n$ of a graph is adjacent with every other vertex then the graph is said to be a *Complete graph.*

### Definition 2.6

For a graph G with n number of vertices, if the vertices are connected by a closed path, then the graph is said to be a $n - cycle$ graph.

### Definition 2.7

A *complete bipartite graph* G is a graph whose vertex set can be subdivided into two subsets say $V_1$ and $V_2$ where no two vertices are adjacent in the same subset, and every pair of vertices$(v_1, v_2)$ where $v_1 \in V_1\ and\ v_2 \in V_2$, are adjacent.

## MAIN RESULTS

### Result 3.1

### Statement

For the distance degree sequence of a graph with $n$ vertices is $(1, (n-1))^n$, then the graph that realizes the distance degree sequence is the Complete Graph.

### Proof

Let the given dds be $(1, (n-1))^n$. Let G be the simple, finite graph that realizes the given dds. There are $n$ vertices in the graph G. From the dds, $'1'$ implies nodes or vertices number at 0 distance from a node say $v_k, k = 1,2, \dots n.$ It is nothing but the same vertex $v_k$. This is because we have G as a simple and finite graph, G has no multiple edges or loops. $(n-1)$ implies the number of nodes or vertices at $'1'$ distance from $v_i$ to any given vertex, that is the vertex $v_i$ is adjacent to all other vertices. We observe all vertices or nodes in G has the identical or same dds. This is evident from fig(ii). This means the graph is a distance degree regular graph. By the definition of a complete graph, every vertex $v_i, i = 1,2, \dots n$ is adjacent to every other vertex. Thus the graph obtained will be a ***complete graph.***

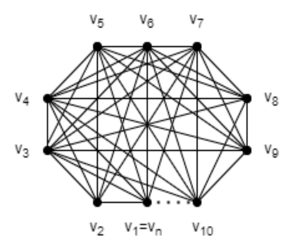


Fig (ii)Graph G

Hence the Result

### Algorithm for Result 3.1

### Requirement

Constructing and Characterizing a graph G with the given Distance Degree Sequence(dds). This graph G is simple and finite.

### Input

The dds $(1, (n-1))^n$ having $n$ number of vertices.

### Initialization

There will be n vertices in the graph G. So initialize the n vertices $v_1, v_2, \dots, v_n$.

The dds $(1, (n-1))^n$ implies the number of nodes or vertices at distance 0 is 1, that is, the vertex $v_i$ itself for every $i = 1,2, \dots n$ and the nodes number at 1 distance is $(n-1)$ for every vertex $v_i, i = 1,2, \dots n.$

### Iterations

#### Step 1

Initialize the vertices $v_1, v_2, \dots v_n$. Ensure the dds of every vertex $v_i, i = 1,2,\dots n$ must be equal to the order of the graph. If TRUE, proceed to next step. Else, proceed to step 5.

#### Step 2

The number of vertices or nodes at distance 0 will be the vertex $v_i, i = 1,2,\dots n$ itself which is trivial as G is a simple, finite graph.

#### Step 3

The number of nodes at distance 1 is $(n-1)$. If the distance between any two vertices $v_i$ *and* $v_j$ for $i \neq j$ is 1, then $v_i$ *and* $v_j$ are adjacent. Here the nodes number at 1 distance is $(n-1)$ which means the vertex $v_i$ is adjacent to every other vertex of the graph as there are $n$ vertices in G.

#### Step 4

Combining step 2 and step 3, we obtain a regular graph. But by step 3, we further claim that the graph G is complete.

#### Step 5

End the computation.

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### Proof of correctness

We prove the algorithm's proof of correctness using Mathematical Induction.

Let G(n) be the simple, finite graph realizing the dds $(1,(n-1))^n$. Then G(n) is a Complete graph.

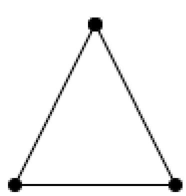
### Proof

We prove the algorithm for all the $n$ vertices.

Base case:

When n = 3, the dds is $(1, 2)^3$. The graph that realizes the DDS is the complete graph with 3 vertices. (1)

The number of vertices is 3. Number of nodes at distance 0 is 1 and the number of vertices or nodes at distance 1 is 2.



Fig(iii) G(3)

From fig(iii), the graph that realizes the dds $(1, 2)^3$ is the complete graph with 3 vertices.

Induction step:

Suppose (1) holds for $n = n-1$. Then the DDS is $(1,(n-2))^{n-1}$. The graph that realizes the DDS is the complete graph with $n-1$ vertices from fig(iv).

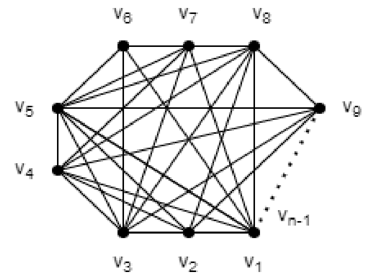
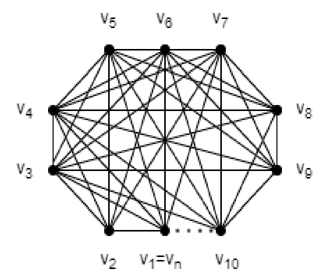



Fig (iv) G(n-1)

We now consider the dds $(1,(n-1))^n$. By hypothesis, the algorithm holds for $n-1$ vertices. Adding one more vertex to the hypothesis, the dds is $(1,(n-1))^n$ as in fig(v). This again results in the complete graph.




Fig(v) G(n)

Hence the result

### Result 3.2

#### Statement

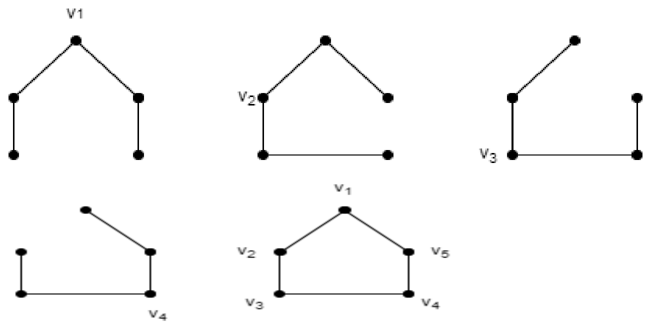
1. Given the Distance Degree Sequence as $(1,2,2,\dots,2)^n$ then the graph that realizes the dds is a $n$ odd cycle graph, $n - \textit{number of vertices of the graph}$.
2. Given the Distance Degree Sequence as $(1,2,2,\dots,2,1)^n$ then the graph that realizes the dds is a $n$ even cycle graph, $n - \textit{number of vertices of the graph}$.

### Proof

(i) Consider the given dds as $(1,2,2,\dots,2)^n$. Let G be the simple, finite graph that realizes the distance degree sequence. There are $n$ vertices in the graph G. $(1,2,2,\dots,2)^n$ implies the number of vertices or nodes at distance $0,1,2,\dots$ are $1,2,2,\dots,2$ respectively for $n$ vertices. Suppose the vertex set be $\{v_1, v_2, \dots v_n\}$. We observe that all $n$ vertices have the same dds. So the graph that realizes the DDS $(1,2,2,\dots,2)^n$ must be a regular graph. Since G is simple, finite graph the number of nodes at 0 distance is the vertex $v_i$ itself for $i = 1,2,\dots n$. The number of nodes or or vertices at distance 1,2,3,…will be 2,2,…2,1 respectively.

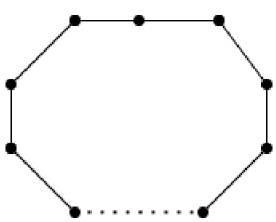
Suppose we consider the dds (1,2,2) for $n = 5$. Then from fig(vi), all five vertices will have the same or identical dds.

We proceed the construction as follows:



Fig(vi) G – 5 oddcycle

Combining the distance degree sequence of vertices $v_1, v_2, v_3, v_4$ we obtain the last figure of the above illustration. So, in general the graph that realizes the distance degree sequence $(1,2,2,\ldots,2)^n$ for $n$ vertices is the $n - odd\ cycle\ graph\ G$. Graph is also regular by the definition of Cycle Graph. Thus, the fig(vii) is obtained.


Fig(vii) G – n odd cycle

(ii) Consider the given dds as $(1,2,2,\ldots,2,1)^n$. Let G be the simple, finite graph that realizes the distance degree sequence. There are $n$ vertices in the graph G. $(1,2,2,\ldots,2)^n$ implies the number of vertices or nodes at distance 0,1,2, ... are 1,2,2, ... ,2,1 respectively for $n$ vertices. Suppose the vertex set be $\{v_1, v_2, \ldots v_n\}$. We observe that all $n$ vertices have the same DDS. So the graph that realizes the DDS $(1,2,2,\ldots,2,1)^n$ must be a regular graph. Since G is simple, finite graph the number of nodes at 0 distance is $v_i$ itself for $i = 1,2,\ldots n$. The number of nodes or vertices at distance 1,2,3,…will be 2,2,…2,1 respectively.

Suppose we consider the dds (1,2,2,1) for $n = 6$ as the sum of the distance degree sequence of every vertex is the number of nodes of the graph. All six vertices will have the identical dds.

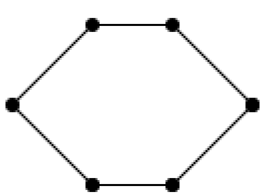


Fig (viii) G – 6 even cycle.

Combining the dds of vertices $v_1, v_2, v_3, v_4, v_5, v_6$. The graph obtained thus is shown in fig(viii). So, in general the graph that realizes the dds $(1,2,2,\ldots,2,1)^n$ for $n$ vertices is the $n - even\ cycle\ graph\ G$. Graph is also regular by the definition of Cycle Graph.

Hence the result.

### Algorithm for Result 3.2

#### Requirement

Constructing and characterizing a graph G with the given dds. This graph G is simple and finite.

#### Input

(i) The dds $(1,2,2,\ldots,2,1)^n$.
(ii) The dds $(1,2,2,\ldots,2,1)^n$.

#### Initialization

There will be n vertices in the graph G. So initialize the n vertices $v_1, v_2, \ldots, v_n$.

#### Iterations

#### Step 1

Ensure the distance degree sequence of every node will equal the order of the graph. If TRUE, proceed to next step. Else proceed to step 9.

#### Step 2

1 in the dds is the number of nodes at 0 distance from vertex $v_i, i = 1,2,\ldots n$. Since G is a simple, finite graph where there are no multiple edges and loops, it will be the vertex $v_i$ itself.

#### Step 3

The number of nodes at distance 1 are 2. So make two vertices adjacent to $v_i$, $i = 1,2,\ldots n$ so that the distance between $v_i$ and the two vertices is 1.

#### Step 4

The number of nodes at distance 2 are 2. So make two vertices adjacent to $v_i$, $i = 1,2,\ldots n$ so that the distance between $v_i$ and the two vertices is 2.

#### Step 5

Repeat step 4 if the nodes number at distance 3, 4,… is 2. If the vertex number that is number of nodes at maximum distance k from any vertex $v_i$ is 2, then proceed to step 7.

#### Step 6

Repeat step 4 if the number of nodes or vertices at distance 3, 4, … is 2. If the number of nodes at maximum distance k from any vertex $v_i$ is 1, then proceed to step 8.

#### Step 7

We obtain an $n - odd\ cycle\ graph\ G$ that realizes the dds $(1,2,2,\ldots,2)^n$ after step 5. G is also regular by the definition of Cycle graph.

#### Step 8

We obtain an $n - even\ cycle\ graph\ G$ that realizes the dds $(1,2,2,\ldots,2,1)^n$ after step 6. G is also regular by the definition of Cycle graph.

### Step 9

End the computation.

### Proof of correctness

The proof for this algorithm can be verified easily using examples for odd and even cycles as illustrated in the proof of the main result 3.2.

### Result 3.3

### Statement

Let the distance degree sequence of a graph be $((1,m,1)^2,(1,2,(m-1))^m)$. The graph that realizes the dds is the complete bipartite $K_{2,m}$ graph. The total number of vertices of $G(K_{2,m})$ is $m+2=n$ vertices.

### Proof

Let the dds of G be $((1,m,1)^2,(1,2,(m-1))^m)$. There must be $m+2$ vertices if there is a graph G. Of the $m+2$ vertices, 2 vertices must have the dds $(1,m,1)$. Remaining $m$ vertices must have the dds $(1,2,(m-1))$.

For the dds $(1,m,1)$;

1 implies the number of nodes at 0 distance

m implies the number of vertices o nodes at 1distance,

1 implies the number of nodes at 2 distance.

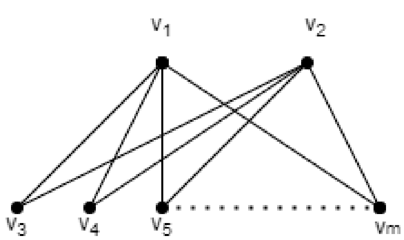
Similarly, for the dds $1,2,(m-1)$:

1 implies the number of vertices or nodes at distance 0,

2 implies the number of nodes at distance 1,

$m-1$ implies the number of nodes at distance 2.

There could be a possibility of existence of bipartite graph as a vertex $v_i$ is adjacent to $m$ vertices but not to all $m+1$ vertices. This implies two vertices must be in one vertex partition say $V_1$. Again for another vertex partition, any vertex $v_i$ must be adjacent to 2 vertices. These two vertices must be the vertices of partition $V_1$. Hence there exists a vertex partition $V_1$ *and* $V_2$. Also every vertex from $V_1$ is adjacent to every vertex from $V_2$ and vice versa. But no vertex is adjacent to another vertex that belongs to the same vertex partition. So, this is a complete bipartite graph $K_{2,m}$. The given dds realizes the complete bipartite graph $G(K_{2,m})$ as shown in fig(ix).




Fig(ix) ***Graph*** $\boldsymbol{G(K_{2,m})}$

Hence the result.

### Comparison of the above Discussed Results with the Existing Work

In the previous works, distance degree sequence of graphs were determined for some derived graphs, like powers of a graph, the total graph, the line graph etc[12]. Also the existence of distance degree regular, distance degree injective and uniform distance distribution in lexicographic product and strong product of graphs have been discussed by Sneha Elsa Idiculla, Revathy N V and Chithra M R[13]. In these works, the dds was determined for a given graph. But in my work, for a given distance degree sequence, the graphs were constructed and characterized to realize the given dds. The proposed Distance Degree sequence talks about every single vertex in a graph. That is, for every single vertex, its distance degree sequence can be found. These sequences can be enumerated or listed as a larger sequence for the whole graph. Due to this property, many characterization results can be constructed for various graphs in a precise manner. But in other sequences especially the degree sequence and eccentric sequence, the degrees and eccentricities of the vertices are listed as the degree sequence and eccentric sequence respectively. Characterization of graphs based on these sequences is quite difficult compared to the distance degree sequence of a graph.

## Conclusion

Here in this article, we have discussed the construction and characterization of complete graphs and n odd and even cycles when the distance degree sequence is given. That is, given a sequence, a graph will exist that realizes the sequence. By doing so, we have proved that the sequence considered is graphical. Many researchers have developed this concept of Distance Degree Sequence and various results have been computed that find application in day to day practical life. This has lead to many open problems. In future, such problems can be discussed and their properties can be computed.

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